

Perturbative evaluation of scalar two-point function in the Cosmic Microwave Background power spectrum

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Abstract

Recent work in the literature has found a suppression or, instead, an enhancement of the Cosmic Microwave Background power spectrum in quantum gravity, although the effect is too small to be observed, in both cases. The present paper studies in detail the equations recently proposed for a Born-Oppenheimer-type analysis of the problem. By using a perturbative approach to the analysis of the nonlinear ordinary differential equation obeyed by the two-point function for scalar fluctuations, we find various explicit forms of such a two-point function, with the associated power spectrum. In particular, a new family of power spectra is obtained and studied. The theoretical prediction of power enhancement at large scales is hence confirmed.

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I. INTRODUCTION

The attempts of building a quantum theory of gravity have given rise, along the years, to substantial theoretical developments, e.g. the discovery of ghost fields in the functional integral [1–3], Hawking radiation [4, 5] and quantum field theory in curved spacetime [6], the background-field method [7, 8], among the many. Quantum gravity is currently expected to unify both the guiding principles and all fundamental interactions of physics, although no agreement exists on whether one should use field-theoretic or, instead, sharply different structures (e.g. strings, branes, twistors, loops and spinfoams). Many calculations in quantum gravity are very detailed and predictive, but unfortunately the length scales and energies involved remain inaccessible to experiments in the laboratories on earth, even though some encouraging evidence exists that we might be approaching the era of quantum gravity phenomenology [9].

Over the last decades, however, the exciting (or puzzling) discoveries in observational cosmology (e.g. dark matter, dark radiation, dark energy, the Cosmic Microwave Background (hereafter CMB) anisotropy spectrum) have led to several theoretical efforts, including the attempt of evaluating the effect of quantum gravity on the CMB power spectrum. Interestingly, the work in Ref. [10] found that, in canonical quantum gravity, a Jeffreys-Wentzel-Kramers-Brillouin analysis of the Wheeler-DeWitt equation can yield a suppression of power at large scales, in a model where a massive scalar field ϕ is coupled to a spatially flat Friedmann-Lemaître-Robertson-Walker universe, and perturbations of ϕ are later considered. Another analysis of the same set of nonlinear equations led in Ref. [11] to the opposite prediction, i.e. an enhanced CMB power spectrum at large scales. Interestingly, a detailed application of Born-Oppenheimer methods [12] to the same problem has led, more recently, to calculations predicting again an enhanced power spectrum at large scales [13]. Since the analysis in Ref. [13] avoids, by construction, all possible inconsistencies related to unitarity violation [12], it has been our aim to gain a deeper understanding of the potentialities of the algorithm developed in Ref. [13].

Section II summarizes recent results obtained from the Born-Oppenheimer technique [14] underlying the analysis in Ref. [13]. Section III studies the homogeneous equation associated to the nonlinear ordinary differential equation obeyed by the two-point function for scalar fluctuations. The complete equation is studied in Sec. IV by means of a perturbative ansatz.

The resulting power spectrum is displayed and discussed in Sec. V. Concluding remarks and open problems are presented in Sec. VI.

II. BRIEF OUTLINE OF RECENT RESULTS OBTAINED FROM THE BORN-OPPENHEIMER METHOD

Following the work in Refs. [10, 11], we study a quantum cosmological model where a real-valued massive scalar field is coupled to gravity in a spatially flat Friedmann-Lemaitre-Robertson-Walker universe with scale factor a . Eventually, as shown in Ref. [13], on considering the function ρ which solves the Ermakov-Pinney equation [15–18]

$$\left(\frac{d^2}{d\eta^2} + \omega^2\right)\rho = \rho^{-3}, \quad (2.1)$$

where η is the conformal-time variable such that $d\eta = \frac{dt}{a}$, one arrives at building the normalized vacuum state (the prime denoting derivative with respect to η)

$$(\pi\rho^2)^{-\frac{1}{4}} \exp\left[\frac{i}{2} \int^\eta \frac{d\tilde{\eta}}{\rho^2} - \frac{\nu^2}{2} \left(\frac{1}{\rho^2} - i\frac{\rho'}{\rho}\right)\right],$$

and one derives a differential equation for the 2-point function $p(\eta)$ describing the spectrum of scalar fluctuations. Such an equation reads as

$$\left[\frac{d^3}{d\eta^3} + 4\omega^2 \frac{d}{d\eta} + 2\frac{d\omega^2}{d\eta}\right]p + \frac{F(\eta)}{m_P^2} = 0, \quad (2.2)$$

where $p(\eta)$ pertains to the vacuum state that reduces to the Bunch-Davies vacuum [19] in the short wavelength regime, and F is found to be

$$\begin{aligned} F(\eta) \equiv & -\frac{d^3}{d\eta^3} \left[\frac{(p'^2 + 4\omega^2 p^2 - 1)}{4a'^2} \right] + \frac{d^2}{d\eta^2} \left[\frac{p'(p'^2 + 4\omega^2 p^2 + 1)}{4pa'^2} \right] \\ & + \frac{d}{d\eta} \left\{ \frac{1}{8a'^2 p^2} \left[(1 - 4\omega^2 p^2)^2 + 2p'^2(1 + 4\omega^2 p^2) + p'^4 \right] \right\} \\ & - \frac{\omega\omega'(p'^2 + 4\omega^2 p^2 - 1)}{a'^2}. \end{aligned} \quad (2.3)$$

III. THE HOMOGENEOUS DIFFERENTIAL EQUATION

The homogeneous differential equation associated with Eq. (2.2) is (see appendix A)

$$\mathcal{L}_\omega(p) \equiv \left(\frac{d^3}{d\eta^3} + 4\omega^2 \frac{d}{d\eta} + 2\frac{d\omega^2}{d\eta}\right)p = 0. \quad (3.1)$$

This third-order equation can be solved by writing p in the form

$$p(\eta) = Y^2(\eta), \quad (3.2)$$

and then considering the combination

$$Z(\eta) = Y'' + \omega^2 Y. \quad (3.3)$$

Equation (3.1) then becomes

$$6Y'Z + 2YZ' = 0, \quad \implies \quad ZY^3 = \text{const} = C_0, \quad (3.4)$$

so that one gets the 1-parameter family of second-order ordinary differential equations

$$\boxed{Y^3(Y'' + \omega^2 Y) = C_0.} \quad (3.5)$$

In fact pointing out that

$$p' = 2YY', \quad p'' = 2Y'^2 + 2YY'', \quad p''' = 2YY''' + 6Y'Y'', \quad (3.6)$$

the third-order equation (3.1) is re-expressed as

$$\mathcal{L}_\omega(Y^2) = 2(YY''' + 3Y'Y'' + 4\omega^2 YY' + 2\omega\omega'Y^2) = 0. \quad (3.7)$$

Multiplying both sides of this equation by Y^2 we obtain the equivalent form

$$0 = Y^2 \mathcal{L}_\omega(Y^2) = 2[Y^3Y'' + \omega^2 Y^4]' = 2[Y^3Z]'. \quad (3.8)$$

In other words, $Y^2 \mathcal{L}_\omega(Y^2)$ vanishes and is itself proportional to the derivative of $Y^3(Y'' + \omega^2 Y)$, so that Eq. (3.5) follows easily.

Therefore, as soon as a special choice of $\omega(\eta)$ is made, Eq. (3.5) can be solved for $Y(\eta)$. For example, when the Hubble parameter H is constant, a natural choice for ω^2 is the following [13]:

$$\omega^2 = k^2 \left(1 - \frac{2}{k^2 \eta^2} \right), \quad (3.9)$$

which implies that $\omega \rightarrow k$ as soon as the conformal time goes to infinity. It is then convenient to introduce the new variable $x \equiv -k\eta$ and rescale the constant $C_0 = k^2 B_0$. One finds for $Y(x)$ the following solution (depending on three arbitrary integration constants):

$$Y^2 = \frac{B_0}{C_1} Y_-^2 + C_1 (2C_2 Y_- + Y_+)^2, \quad (3.10)$$

where

$$Y_-(x) = \cos x - \frac{\sin x}{x} = -\sqrt{\frac{\pi x}{2}} J_{3/2}(x), \quad Y_+(x) = \sin x + \frac{\cos x}{x} = \sqrt{\frac{\pi x}{2}} J_{-3/2}(x), \quad (3.11)$$

and only $Y_-(x)$ has a finite limit at $x = 0$. After a suitable redefinition of constants, one finds then for Y^2 the three elementary solutions

$$Y^2 = c_1 Y_+^2 + c_2 Y_-^2 + c_3 Y_+ Y_-. \quad (3.12)$$

This class of solutions is rich enough and contains either combination of Bessel- J functions and polynomials. In fact, when $c_3 = 2\sqrt{c_1 c_2}$, Eq. (3.12) reduces to

$$Y^2 = (\sqrt{c_1} Y_+ + \sqrt{c_2} Y_-)^2. \quad (3.13)$$

Similarly, when $c_3 = 0$ and $c_1 = c_2 = 1$ we have

$$Y^2 = Y_-^2 + Y_+^2 = 1 + \frac{1}{x^2}. \quad (3.14)$$

In particular, following Ref. [13], the choice of constants $c_1 = 1/2$, $c_2 = 0 = c_3$, which implies

$$Y(x) = \frac{1}{\sqrt{2}} Y_+(x), \quad (3.15)$$

should be preferred.

In view of its simplicity and of its relevance for a pure de Sitter expansion [13], in the following perturbative analysis, we will use as solutions of the homogeneous equation (3.1)

$$p(x) = \frac{1}{2k} \left(1 + \frac{1}{x^2} \right) \equiv p_0(x), \quad \omega(x) = k \sqrt{1 - \frac{2}{x^2}} \equiv \omega_0(x), \quad (3.16)$$

where we have restored for convenience the original variables p and ω . In terms of the auxiliary variable defined in Eq. (3.2), such a solution satisfies Eq. (3.5) with $C_0 = \frac{1}{4}$ which is associated, in turn, with the chosen initial data for $Y(x)$ and $\omega(x)$ at $x = 0$; in fact $C_0 = Y^3(0)[Y''(0) + \omega^2(0)Y(0)]$. Our choice of initial conditions results from the request (as in Ref. [13], Eqs. (32) and (33) therein) to reproduce the de Sitter result in absence of quantum corrections.

IV. THE COMPLETE EQUATION

Since it is rather difficult to solve exactly the complete equation (2.5), we now look for some specific conditions that make it possible to find a perturbative solution. In a viable

single-field inflationary model, one has an evolution of cosmological perturbations based on the slow-roll paradigm. However, in order to illustrate the main effect of quantum gravity on the spectrum, it is sufficient to neglect slow-roll parameters, which leads to a pure de Sitter expansion $a(t) = e^{Ht}$ with constant H , for which the condition $\frac{da}{dt} = Ha$, re-expressed through conformal time η , becomes

$$\frac{1}{a} \frac{da}{d\eta} = Ha \implies d\left(\frac{1}{a}\right) = -H d\eta \implies a = -\frac{1}{H\eta}. \quad (4.1)$$

Besides $x \equiv -k\eta$ we also introduce the rescaled variables

$$\Omega \equiv \frac{\omega}{k}, \quad P \equiv kp, \quad (4.2)$$

as well as the (small) quantity

$$\varepsilon \equiv \frac{H^2}{m_P^2 k^3}. \quad (4.3)$$

We undertake now the analysis of solutions which perturb the special one given in Eq. (3.16), that we write in the form

$$P(x) = \frac{1}{2} \left(1 + \frac{1}{x^2}\right) + \varepsilon P_1(x), \quad \Omega^2(x) = 1 - \frac{2}{x^2} + \varepsilon W_1(x), \quad (4.4)$$

where the unperturbed solutions P_0 and Ω_0 have been introduced before, in (3.16). Note that, by using for P_0 either Y_+^2 , Y_-^2 or Y_+Y_- (or a linear combination of them) instead of the simple solution (4.4) results only in some mathematical complications. For example, one finds the cumbersome equation (B1) in appendix B. By using for $P_0(x)$ (i.e., for $Y_0(x)$) and $\Omega_0(x)$ the expressions given in (4.4), Eq. (B1) is much simplified and reduces to

$$-P_1''' - 4 \left(1 - \frac{2}{x^2}\right) P_1' - \left(1 + \frac{1}{x^2}\right) W_1' - \frac{8}{x^3} P_1 + \frac{4}{x^3} (W_1 - 1) = 0. \quad (4.5)$$

A simple inspection of this equation suggests introducing

$$\widetilde{W}_1 \equiv W_1 - 1, \quad (4.6)$$

so that the final equation for perturbative quantities is given by

$$-x^3 P_1''' - 4x(x^2 - 2) P_1' - 8P_1 - x(x^2 + 1) \widetilde{W}_1' + 4\widetilde{W}_1 = 0, \quad (4.7)$$

that is, recalling the definition of the operator \mathcal{L}_Ω of Eq. (3.1)

$$\mathcal{L}_{\Omega_0}(P_1) = - \left(1 + \frac{1}{x^2}\right) \widetilde{W}_1' + \frac{4}{x^3} \widetilde{W}_1 \equiv Q(x), \quad (4.8)$$

$Q(x)$ being our notation for the inhomogeneous term, so that

$$\widetilde{W}_1(x) = \left(\frac{x^2}{1+x^2} \right)^2 \left(C_1 - \int^x \frac{Q(z)(z^2+1)}{z^2} dz \right). \quad (4.9)$$

This equation can be solved formally and the result is as follows:

$$P_1(x) = \int^x Q(z)G(x,z)dz + c_1Y_-^2 + c_2Y_+^2 + c_3Y_+Y_-, \quad (4.10)$$

where the Green function is given by

$$G(x,z) = \frac{1}{2} (Y_+(z)Y_-(x) - Y_+(x)Y_-(z))^2, \quad (4.11)$$

and the last three terms represent the general solution of the associated homogeneous equation. Note that

$$G(x,x) = 0, \quad \partial_x G(x,z)|_{z=x} = 0, \quad \partial_{xx} G(x,z)|_{z=x} = 1. \quad (4.12)$$

Equation (4.10) can be proven as follows (omitting the solution of the homogeneous equation for which it is simply $\mathcal{L}_{\Omega_0}(c_1Y_-^2 + c_2Y_+^2 + c_3Y_+Y_-) = 0$). Using Eqs. (4.12) we have immediately

$$\begin{aligned} \frac{d}{dx}P_1 &= Q(x)G(x,x) + \int^x Q(z)\partial_x G(x,z)dz = \int^x Q(z)\partial_x G(x,z)dz, \\ \frac{d^2}{dx^2}P_1 &= Q(x)\partial_x G(x,z)|_{z=x} + \int^x Q(z)\partial_{xx} G(x,z)dz = \int^x Q(z)\partial_{xx} G(x,z)dz, \\ \frac{d^3}{dx^3}P_1 &= Q(x)\partial_{xx} G(x,z)|_{z=x} + \int^x Q(z)\partial_{xxx} G(x,z)dz \\ &= Q(x) + \int^x Q(z)\partial_{xxx} G(x,z)dz, \end{aligned} \quad (4.13)$$

hence the sought for result

$$\mathcal{L}_{\Omega_0}(P_1) = Q(x) + \int^x Q(z)[\mathcal{L}_{\Omega_0}G(x,z)]dz = Q(x). \quad (4.14)$$

In spite of this nice result for the general representation of P_1 , however, we are going to consider special situations in which $Q(x)$ is given.

For example, we can find a series solution consistently for P_1 and \widetilde{W}_1 . It is worth discussing separately the following simple cases.

- Case $P_1 = 0$. In this case we have

$$-x(x^2+1)\widetilde{W}_1' + 4\widetilde{W}_1 = 0, \quad (4.15)$$

with solution

$$\widetilde{W}_1 = C_1 \left(\frac{x^2}{1+x^2} \right)^2. \quad (4.16)$$

- Case $\widetilde{W}_1 = 0$. In this case we have the homogeneous equation

$$x^3 P_1''' + 4x(x^2 - 2) P_1' + 8P_1 = 0, \quad (4.17)$$

with the known solution

$$P_1 = c_1 Y_-^2 + c_2 Y_+^2 + c_3 Y_+ Y_-. \quad (4.18)$$

- Series solution for both P_1 and \widetilde{W}_1 .

Looking for solutions having the form

$$P_1 = \sum_{k=0}^{n_1} A_k x^k, \quad \widetilde{W}_1 = \sum_{k=0}^{n_2} B_k x^k, \quad (4.19)$$

a particular solution involving a minimum number of coefficients A_k and B_k (“minimal solution”) is given by

$$P_1 = A_0 + A_2 x^2, \quad \widetilde{W}_1 = 2A_0 - 4A_2 x^2, \quad (4.20)$$

with $n_1 = n_2 = 2$ and A_0 and A_2 undetermined constants. Note that, even though in these solutions there appear powers of the time variable x , which imply a more rapid growth of the perturbation itself, there is enough room for the study presented here. In fact, the condition for obtaining perturbations that can be thrust can be expressed by the majorization

$$\left| \varepsilon \frac{P_1}{\widetilde{W}_1} \right| \ll 1,$$

and the very small values of ε (getting smaller at higher wave numbers k) allow anyway for polynomial variations of the time variable, even of degree much larger than 2.

V. THE POWER SPECTRUM

For the 3 cases considered in Sec. IV we can now evaluate the power spectrum \mathcal{P}_ν , given by [13]

$$\mathcal{P}_\nu = \frac{k^3}{2\pi^2} p = \left(\frac{k}{2\pi} \right)^2 2P \equiv \mathcal{P}_* 2P. \quad (5.1)$$

For example, in the 3 perturbative cases considered above we have

- Case $P_1 = 0$

$$\mathcal{P}_\nu = \mathcal{P}_* \left(1 + \frac{1}{x^2} \right). \quad (5.2)$$

- Case $\widetilde{W}_1 = 0$

$$\mathcal{P}_\nu = \mathcal{P}_* \left\{ 1 + \frac{1}{x^2} + 2\varepsilon [c_1 Y_-^2 + c_2 Y_+^2 + c_3 Y_+ Y_-] \right\}. \quad (5.3)$$

The additional term of first order in ε has the following MacLaurin expansion:

$$c_2 \left(1 + \frac{1}{x^2} \right) - \frac{c_3}{3} \left(1 - \frac{2}{5} x^2 \right) x + \mathcal{O}(x^4). \quad (5.4)$$

- Series solution for both P_1 and \widetilde{W}_1 . Last, but not least, the opportunities offered by Eq. (4.20), are richer because, even in the case of the “minimal solution”, the resulting power spectrum depends on a pair of arbitrary constants and reads as

$$\mathcal{P}_\nu = \mathcal{P}_* \left[1 + \frac{1}{k^2 \eta^2} + 2\varepsilon (A_0 + A_2 k^2 \eta^2) \right]. \quad (5.5)$$

Here, the term proportional to A_0 scales as k^{-3} and leads to an increase of power for large scales, as in Ref. [13].

In all previous cases the dependence of the power spectrum on several constants can be used to fit experimental data. Such data, however, are far beyond the actual sensitivity of existing devices. More precisely, in Ref. [11] it has been shown that the quantum-gravitationally corrected Schrödinger equation leads to a modification (to first order in ε) of the power spectrum by a correction function C_k , such that one can translate this modification also to the standard power spectrum in the following way:

$$\mathcal{P}_\nu^{(1)}(k) = \mathcal{P}_\nu^{(0)}(k) C_k^2. \quad (5.6)$$

One can write

$$C_k^2 = 1 + \delta_{\text{WDW}}^\pm(k) + \mathcal{O}(\varepsilon^2), \quad (5.7)$$

where $\delta_{\text{WDW}}^\pm(k)$ either takes the form

$$\delta_{\text{WDW}}^+(k) = 179.09\varepsilon, \quad (5.8)$$

or the form

$$\delta_{\text{WDW}}^-(k) = -247.68\varepsilon. \quad (5.9)$$

Note that we can also cast our result in a form similar to that given by Eq. (5.6). In fact, for a general power spectrum of the form

$$\mathcal{P}_\nu = \mathcal{P}_\nu^{(0)}(k) \left(1 + \varepsilon \frac{k^2 \eta^2}{(1 + k^2 \eta^2)} \mathcal{F}_{(k,\eta)} \right) + \mathcal{O}(\varepsilon^2), \quad (5.10)$$

from which the identification

$$C_k^2 = 1 + \delta(k, \eta) + \mathcal{O}(\varepsilon^2), \quad (5.11)$$

where

$$\delta(k, \eta) \equiv \varepsilon \frac{k^2 \eta^2}{(1 + k^2 \eta^2)} \mathcal{F}_{(k,\eta)}. \quad (5.12)$$

For instance, in the case (5.5) discussed above we have

$$\mathcal{F}_{(k,\eta)} = 2(A_0 + A_2 k^2 \eta^2), \quad (5.13)$$

and then

$$\delta(k, \eta) = \varepsilon \frac{2k^2 \eta^2}{(1 + k^2 \eta^2)} (A_0 + A_2 k^2 \eta^2). \quad (5.14)$$

In order to compare $\delta(k, \eta)$ with its WDW counterparts, say $\delta_{\text{WDW}}(k) = C\varepsilon$ where C is a constant, one can compute for example $\delta(k, \eta)$ at a certain value $k_* \eta_*$ properly chosen (e.g., the value η_* which extremizes $\delta(k, \eta)$); this choice, as well as other similar choices, leads to a $\delta(k_*, \eta_*)$ which still depends upon A_0 and A_2 and such parameters can be adjusted to fit experimental data, for example the (slow-roll) parameters of any inflationary model. It is worth stressing that the factor multiplying ε on the right-hand side of (5.14) plays the role of modulating factor. This concept will be discussed again below.

The basic equations in the theory of the spectral index n_s and its running α_s involve the slow-roll parameters [20] $\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}$, $\epsilon \equiv -\frac{\dot{H}}{H^2} = 2\frac{\dot{\phi}^2}{H^2}$, $\Xi^2 \equiv \frac{1}{H^2} \frac{d}{dt} \frac{\ddot{\phi}}{\dot{\phi}}$, where we have used $8\pi G = 1$ units in the formula for ϵ . Therefore

$$n_s - 1 \equiv \frac{d \log \mathcal{P}_\nu}{d \log k} \approx 2\eta - 4\epsilon - 3\delta_{\text{WDW}}^\pm \quad (5.15)$$

and

$$\alpha_s \equiv \frac{dn_s}{d \log k} \approx 2(5\epsilon\eta - 4\epsilon^2 - \Xi^2) + 9\delta_{\text{WDW}}^\pm, \quad (5.16)$$

where use has been made of the approximate formula

$$\frac{d}{d \log k} \approx \frac{1}{H} \frac{d}{dt}, \quad (5.17)$$

jointly with the equations of motion.

The work in Ref. [11] has shown that the absolute value of $\delta_{\text{WDW}}^{\pm}$ is majorized by numbers of order 10^{-12} when k is replaced by k/k_{min} , where k_{min} is the largest observable scale, or by numbers of order 10^{-9} when k is replaced by k/k_0 , where k_0 is the pivot scale used in the WMAP9 analysis. By comparing such quantum-gravitational corrections to the spectral index n_s and its running α_s derived above with the values determined from the WMAP9 data, $n_s = 0.9608 \pm 0.0080$ and $\alpha_s = -0.023 \pm 0.011$ (using the WMAP9+eCMB+BAO+ H_0 dataset in both cases) [21], and the 2013 results of the Planck mission, $n_s = 0.9603 \pm 0.0073$ and $\alpha_s = -0.013 \pm 0.009$ (using additionally the WMAP polarization data in both cases) [22], one sees that the corrections in [11] are completely drowned out by the statistical uncertainty in the data. We are currently trying to understand whether one can arrive at formulas where δ_{WDW} is systematically replaced by our $\delta(k, \eta)$.

A. Special solutions of the perturbative equation

It is easy to find special solutions to the final perturbative equation with more terms. For example, looking for a sixth-order polynomial solution of Eq. (4.7)

$$P_1^{(6)} = \frac{1}{2}B_0 + \left(\frac{5}{4}B_5 - \frac{3}{8}B_3\right)x - \frac{1}{4}B_2x^2 + \left(-\frac{1}{4}B_3 + \frac{1}{2}B_5\right)x^3 + \left(-\frac{1}{4}B_4 + \frac{9}{8}B_6\right)x^4 - \frac{1}{4}B_5x^5 - \frac{1}{4}B_6x^6, \quad (5.18)$$

$$\widetilde{W}_1^{(6)} = B_0 + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6, \quad (5.19)$$

where the B_k are arbitrary constants, we have

$$P(x) = \frac{1}{2} \left(1 + \frac{1}{x^2}\right) + \varepsilon P_1^{(6)}(x), \quad \Omega^2(x) = 1 - \frac{2}{x^2} + \varepsilon(1 + \widetilde{W}_1^{(6)}(x)), \quad (5.20)$$

and the resulting power spectrum is given by

$$\frac{\mathcal{P}_\nu}{\mathcal{P}_*} = 1 + \frac{1}{x^2} + 2\varepsilon P_1^{(6)}(x). \quad (5.21)$$

Another (perhaps more interesting) solution should involve also negative powers of x . For this purpose, we look for solutions in the form

$$P_1^{(-3,3)} = \sum_{k=-3}^3 A_k x^k, \quad \widetilde{W}_1^{(-3,3)} = \sum_{k=-3}^3 B_k x^k, \quad (5.22)$$

where there is full freedom to choose the lower and upper summation limits. For simplicity, we have here set the limits equal to -3 and 3 , respectively. By inserting such an ansatz into Eq. (4.7) we find

$$\begin{aligned} P_1^{(-3,3)} &= -\frac{1}{4} \frac{B_{-3}}{x^3} - \frac{1}{2} \frac{B_0}{x^2} - \frac{1}{4} \left(B_1 + \frac{3}{2} B_3 \right) x - \frac{B_2}{4} x^2 - \frac{B_3}{4} x^3, \\ \widetilde{W}_1^{(-3,3)} &= \frac{B_{-3}}{x^3} - \frac{B_1}{x} + B_0 + B_1 x + B_2 x^2 + B_3 x^3, \end{aligned} \quad (5.23)$$

where compatibility requires that $B_{-1} = -B_1$ and A_0 can be always set to 0. An interesting particular case of such a framework corresponds to choosing $B_1 = B_2 = B_3 = 0$. Our previous formula reduces then to

$$\begin{aligned} P_1^{(-3,3)} &= -\frac{1}{4} \frac{B_{-3}}{x^3} - \frac{1}{2} \frac{B_0}{x^2}, \\ \widetilde{W}_1^{(-3,3)} &= \frac{B_{-3}}{x^3} - \frac{B_1}{x} + B_0. \end{aligned} \quad (5.24)$$

Note that the result (33) in Ref. [13] is described by the simple choice $B_{-3} = 0, B_0 = -2$.

Our power spectrum can be written in the form

$$\frac{\mathcal{P}_\nu}{\mathcal{P}_*} = 1 + \frac{1}{x^2} - \frac{\varepsilon}{x^2} \left(\frac{B_{-3}}{x} + B_0 \right). \quad (5.25)$$

This implies, with the notation used in (5.7), that

$$\delta_{\text{WDW}} = -\varepsilon \frac{(B_{-3} + B_0 x)}{x(x^2 + 1)}. \quad (5.26)$$

As we have said earlier, it is interesting to look at the structure of the modulating factor in the formula expressing the enhancement of the power spectrum, i.e., the ratio $\frac{\delta_{\text{WDW}}}{\varepsilon}$ in the above formula. A nice feature of such a ratio is its summability on the whole real line, provided one adopts the principal-value prescription for the integral including the origin. With this understanding, one can evaluate its average, which is then equal to πB_0 . This makes it possible to compare straight away our $\langle \delta_{\text{WDW}} \rangle$ with other values, whether or not existing in the literature.

VI. CONCLUDING REMARKS AND OPEN PROBLEMS

In our paper we have applied a perturbative technique for the evaluation of the scalar two-point function in the CMB power spectrum, relying upon the general technique developed

in Ref. [13] for de Sitter evolution, with the associated fundamental equations (2.2) and (2.3). Our results in Sec. IV, elegant and at the same time simple in their derivation, are entirely original and lead to the theoretical formulas for the power spectrum displayed in Sec. V which may depend on several parameters. Such formulas reduce to the existing ones [13] in a particular case, but have better potentialities because the lower and upper limit of summation in (5.22) are arbitrary. Hence one might arrive at more accurate theoretical predictions, to be hopefully checked against observations.

Recently, the work in Ref. [23], relying upon Ref. [13], has evaluated the spectra of scalar and tensor perturbations to first order in the slow-roll approximation, which has been found to provide qualitatively new quantum gravitational effects with respect to the pure de Sitter case. We think it would also be interesting to apply our technique to the slow-roll phase studied therein.

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Appendix A: The operator \mathcal{L}_ω

The linear differential operator defined in Eq. (3.1) is not a derivation and hence does not obey the Leibniz rule, but satisfies instead the following property:

$$\mathcal{L}_\omega(fg) = f\mathcal{L}_\omega(g) + g\mathcal{L}_\omega(f) + 3(f'g')' - 2(\omega^2)'fg. \quad (\text{A1})$$

In the case $f = g$ this relation implies

$$\mathcal{L}_\omega(f^2) = 2f\mathcal{L}_\omega(f) + 3(f'^2)' - 4\omega\omega'f^2. \quad (\text{A2})$$

Another useful relation which follows from those considered above states that

$$\mathcal{L}_\omega((f - g)^2) = 2(f - g)[\mathcal{L}_\omega(f) - \mathcal{L}_\omega(g)] + 6(f' - g')(f'' - g'') - 4\omega\omega'(f - g)^2. \quad (\text{A3})$$

Appendix B: Perturbations of solutions of the homogeneous equation; the composite function $F(\eta(x))$

The cumbersome equation mentioned after Eq. (4.4) reads, explicitly, as

$$\begin{aligned}
& \frac{d^3}{dx^3}P_1(x) = -4P_1(x)\Omega_0(x)\frac{d}{dx}\Omega_0(x) \\
& + \left[-4Y_0^2(x)\left(\frac{d}{dx}\Omega_0(x)\right) - 16\Omega_0(x)Y_0(x)\left(\frac{d}{dx}Y_0(x)\right) \right] \Omega_1(x) \\
& - 4\Omega_0^2(x)\left(\frac{d}{dx}P_1(x)\right) + \left(\frac{x^4}{Y_0^3(x)}(1-4C_0) + 48x^2Y_0(x)\right)\left(\frac{d}{dx}Y_0(x)\right)^3 \\
& + \left[-16x^3Y_0^2(x)\Omega_0^2(x) + 20x^4Y_0^2(x)\Omega_0(x)\left(\frac{d}{dx}\Omega_0(x)\right) + 24xY_0^2(x) \right] \left(\frac{d}{dx}Y_0(x)\right)^2 \\
& + \left[112x^3Y_0^3(x)\Omega_0(x)\left(\frac{d}{dx}\Omega_0(x)\right) + \frac{x^4}{Y_0^5(x)}\left(\frac{1}{2} - 12C_0^2\right) \right. \\
& + \frac{x^4\Omega_0^2(x)}{Y_0(x)} + 16x^4Y_0^3(x)\Omega_0(x)\left(\frac{d^2}{dx^2}\Omega_0(x)\right) - 4\frac{x^4\Omega_0^2(x)C_0}{Y_0(x)} + 72\frac{x^2C_0}{Y_0(x)} \\
& \left. - \frac{6x^2}{Y_0(x)} + 48x^2Y_0^3(x)\Omega_0^2(x) + 16x^4Y_0^3(x)\left(\frac{d}{dx}\Omega_0(x)\right)^2 + \frac{x^4C_0}{Y_0^5(x)} \right] \left(\frac{d}{dx}Y_0(x)\right) \\
& - 4Y_0^2(x)\Omega_0(x)\left(\frac{d}{dx}\Omega_1(x)\right) + 24\frac{x^3C_0^2}{Y_0^4(x)} - 4\frac{x^3C_0}{Y_0^4(x)} \\
& - 16x^3C_0\Omega_0^2(x) - 6x + 24xY_0^4(x)\Omega_0^2(x) \\
& + 6x^4Y_0^4(x)\left(\frac{d}{dx}\Omega_0(x)\right)\left(\frac{d^2}{dx^2}\Omega_0(x)\right) + 8x^3\Omega_0^2(x) - \frac{1}{2}\frac{x^3}{Y_0^4(x)} \\
& - 12x^4Y_0^4(x)\Omega_0^3(x)\left(\frac{d}{dx}\Omega_0(x)\right) - 16x^3Y_0^4(x)\Omega_0^4(x) \\
& + 24x^3Y_0^4(x)\Omega_0(x)\left(\frac{d^2}{dx^2}\Omega_0(x)\right) + 72x^2Y_0^4(x)\Omega_0(x)\left(\frac{d}{dx}\Omega_0(x)\right) \\
& + 24x^3Y_0^4(x)\left(\frac{d}{dx}\Omega_0(x)\right)^2 + 2x^4Y_0^4(x)\Omega_0(x)\left(\frac{d^3}{dx^3}\Omega_0(x)\right) \\
& + 2x^4\Omega_0(x)\left(\frac{d}{dx}\Omega_0(x)\right) + 4C_0x^4\Omega_0(x)\left(\frac{d}{dx}\Omega_0(x)\right), \tag{B1}
\end{aligned}$$

where $P_0(x) = Y_0^2(x)$ and $\Omega_0(x)$ are generic solutions of the homogeneous equation as studied in Sec. III.

If P_1 vanishes, the composite function $F(\eta(x))$ is obtained from the general formulae (2.3), (4.1)–(4.4) through the formula

$$F(\eta(x))[P_1 = 0] = \varepsilon m_P^2 k^2 \tilde{F}(\eta(x))[P_1 = 0] + O(\varepsilon^2), \tag{B2}$$

having set

$$\begin{aligned}\tilde{F}(\eta(x))[P_1 = 0] &\equiv \left\{ \frac{6}{x^5} + \frac{1}{2} \frac{d^2}{dx^2} \left[(1+x^2)^{-1} (-x^{-3} - 3x^{-1} + 2x^3) \right] \right. \\ &\quad - \frac{1}{2} \frac{d}{dx} \left[(1+x^2)^{-2} (x^{-4} (2+3x^2)^2 + 2x^2 (2-3x^{-4} - 2x^{-6}) + x^{-4}) \right] \\ &\quad \left. - \frac{2(1+3x^2)}{x^5} \right\},\end{aligned}\tag{B3}$$

where

$$\frac{d^2}{dx^2} \left[(1+x^2)^{-1} (-x^{-3} - 3x^{-1} + 2x^3) \right] = (1+x^2)^{-3} \left[-12x^{-5} - 40x^{-3} - 48x^{-1} - 24x - 4x^3 \right],\tag{B4}$$

$$\begin{aligned}& - \frac{d}{dx} \left[(1+x^2)^{-2} (x^{-4} (2+3x^2)^2 + 2x^2 (2-3x^{-4} - 2x^{-6}) + x^{-4}) \right] \\ &= (1+x^2)^{-3} \left[4x^{-5} + 20x^{-3} + 36x^{-1} + 28x + 8x^3 \right],\end{aligned}\tag{B5}$$

which lead to

$$F(\eta(x))[P_1 = 0] = -\frac{4\epsilon m_P^2 k^2}{x^3} + O(\epsilon^2).\tag{B6}$$

Interestingly, we also find the simple but nontrivial equality

$$\tilde{F}(\eta(x))[P_1 = 0] = \tilde{F}(\eta(x))[W_1 = 0] = \tilde{F}(\eta(x))[P_1 \neq 0, W_1 \neq 0],\tag{B7}$$

which means that, to first order in ϵ , the value taken by $F(\eta(x))$ is unaffected by P_1 and W_1 .

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